J. Fluid Mech. (1968), *vol.* 31, *part* 3, *pp.* 449–453
Printed in Great Britain

A note on shock dynamics relative to a moving frame

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(Received 16 March 1967)

In this paper it is shown how a previous theory for treating shocks moving into fluid at rest (Whitham 1957, 1959) can be changed to a form suitable for treating propagation into a steady uniform flow. It is necessary only to transform the original theory to a moving frame but the details are not trivial.

1. Introduction

In order to study problems of shocks propagating into an existing uniform flow, Chisnell (1965) proposed an extension of a method developed for cases where the fluid ahead is at rest. He studied the problem directly and made further assumptions about the gas dynamics involved. However, since the two problems must preserve Galilean invariance, it is only necessary to transform the problem which has the flow to a frame of reference moving with that flow, and then apply the original theory. When this is done, the results do not agree with Chisnell's. In fact there is one crucial qualitative difference: the rays are no longer orthogonal to the wave fronts. This is well known in geometrical acoustics and one expects a similar result in the non-linear theory. The appropriate transformation is given in §2, and in §3 it is shown that the non-linear theory reduces correctly to geometric acoustics in the limit of weak shocks.

2. Transformation between moving frames

Let frame I with co-ordinates (x', y', t') be one in which the medium ahead of the shock is at rest. Let frame II have co-ordinates (x, y, t) where

$$x = x' + Ut', \quad y = y', \quad t = t'. \tag{1}$$

In frame II the shock appears to be moving through a uniform flow whose velocity is U. Let a_0 be the sound speed of the fluid.

In frame I the shock position is described by a function $\alpha'(x', y')$ such that

$$a_0 t' = \alpha'(x', y') \tag{2}$$

is the shock position at time t'. The shock velocity is normal to it and has magnitude $a_0 M'$ where

$$M' = (\alpha'^2_{x'} + \alpha'^2_{y'})^{-\frac{1}{2}}. \tag{3}$$

Similarly in frame II the shock is described by

$$a_0 t = \alpha(x, y). \tag{4}$$

We can relate the two descriptions through the transformation (1). From (1) and (2) the shock may be described in frame II by the equation

$$a_0 t = \alpha'(x - Ut, y). \tag{5}$$

Hence $\alpha(x, y)$ is the solution of

$$\alpha = \alpha'(x - m\alpha, y), \tag{6}$$

where

$$m = U/a_0. \tag{7}$$

In frame I, the theory (Whitham 1959) proposes that the shock will move so that $\alpha'(x', y')$ satisfies the equation

$$\frac{\partial}{\partial x'}\left(\frac{M'}{A'}\frac{\partial \alpha'}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(\frac{M'}{A'}\frac{\partial \alpha'}{\partial y'}\right) = 0, \tag{8}$$

where

$$M' = (\alpha'^2_{x'} + \alpha'^2_{y'})^{-\frac{1}{2}}, \tag{9}$$

$$A' = f(M'), \tag{10}$$

and $f(M')$ is a prescribed function. We can also satisfy (9) and (8) by the representation

$$\left.\begin{aligned} M'\frac{\partial \alpha'}{\partial x'} &= \cos\theta', & M'\frac{\partial \alpha'}{\partial y'} &= \sin\theta', \\ \cos\theta' &= A'\frac{\partial \beta'}{\partial y'}, & \sin\theta' &= -A'\frac{\partial \beta'}{\partial x'}. \end{aligned}\right\} \tag{11}$$

The curves $\beta' =$ constant are orthogonal to the shock positions $\alpha' =$ constant. We may choose (α', β') as co-ordinates in place of (x', y'). Then, eliminating (x', y') from (11), we have

$$\left.\begin{aligned} \frac{\partial \theta'}{\partial \alpha'} + \frac{1}{A'}\frac{\partial M'}{\partial \beta'} &= 0, \\ \frac{\partial \theta'}{\partial \beta'} - \frac{1}{M'}\frac{\partial A'}{\partial \alpha'} &= 0. \end{aligned}\right\} \tag{12}$$

These can be interpreted geometrically as discussed in an earlier paper (Whitham 1957).

To transfer to frame II, we transform the equations (8), (9), (10) for $\alpha'(x', y')$ into equations for the function $\alpha(x, y)$ defined in (6). This part is now purely mathematical. We have a partial differential equation for $\alpha'(x', y')$ and we introduce a function $\alpha(x, y)$ defined as the solution of

$$\alpha = \alpha'(x - m\alpha, y).$$

Thus any differential operations on $\alpha'(x', y')$ transform according to

$$\left.\begin{aligned} \frac{\partial}{\partial x'} &= \frac{1}{1 - m\alpha_x}\frac{\partial}{\partial x}, \\ \frac{\partial}{\partial y'} &= \frac{m\alpha_y}{1 - m\alpha_x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}; \end{aligned}\right\} \tag{13}$$

in particular
$$\alpha'_{x'} = \frac{\alpha_x}{1-m\alpha_x}, \quad \alpha'_{y'} = \frac{\alpha_y}{1-m\alpha_x}. \tag{14}$$

The differential equation (8) becomes

$$\frac{\partial}{\partial x}\left(\frac{M'}{A'}\frac{\alpha_x}{1-m\alpha_x}\right) + m\alpha_y \frac{\partial}{\partial x}\left(\frac{M'}{A'}\frac{\alpha_y}{1-m\alpha_x}\right) + (1-m\alpha_x)\frac{\partial}{\partial y}\left(\frac{M'}{A'}\frac{\alpha_y}{1-m\alpha_x}\right) = 0, \tag{15}$$

where
$$M' = \frac{1-m\alpha_x}{(\alpha_x^2+\alpha_y^2)^{\frac{1}{2}}}, \quad A' = f(M'). \tag{16}$$

The basic geometry in this theory is in terms of rays and ray tubes formed by bundles of neighbouring rays. If $\mathbf{i}$ is a unit vector along a ray, A is the cross-sectional area of a slender ray tube and V is the volume of a slender ray tube between two successive shock positions, then

$$\begin{aligned}\int_V \nabla \cdot \left(\frac{\mathbf{i}}{A}\right) dV &= \int_{S_2} \frac{\mathbf{n}.\mathbf{i}}{A} dS - \int_{S_1} \frac{\mathbf{n}.\mathbf{i}}{A} dS \\ &= \left[\frac{\mathbf{n}.\mathbf{i}}{A} S\right]_1^2.\end{aligned} \tag{17}$$

Here S_1 and S_2 are the portions of the successive shock surfaces which make up the end faces of V, and $\mathbf{n}$ is the normal to the shock surfaces.

Now, even if the rays are not normal to the shocks, $\mathbf{n}.\mathbf{i}S = A$; therefore, (17) is zero and

$$\nabla . (\mathbf{i}/A) = 0. \tag{18}$$

So we look for a divergence form of (15) and find that it does indeed take this form provided

$$A = \frac{A'}{(1+m^2\alpha_y^2)^{\frac{1}{2}}}, \tag{19}$$

$$\mathbf{i} = \frac{(\alpha_x + m\alpha_y^2,\ \alpha_y - m\alpha_x\alpha_y)}{[(\alpha_x^2+\alpha_y^2)(1+m^2\alpha_y^2)]^{\frac{1}{2}}}. \tag{20}$$

Since
$$\mathbf{n} = \frac{(\alpha_x, \alpha_y)}{(\alpha_x^2+\alpha_y^2)^{\frac{1}{2}}}, \tag{21}$$

we have
$$\begin{aligned}\mathbf{i}.\mathbf{n} &= \frac{\alpha_x(\alpha_x + m\alpha_y^2) + \alpha_y(\alpha_y - m\alpha_x\alpha_y)}{(\alpha_x^2+\alpha_y^2)(1+m^2\alpha_y^2)^{\frac{1}{2}}} \\ &= (1+m^2\alpha_y^2)^{-\frac{1}{2}}.\end{aligned} \tag{22}$$

We see then, that the rays are not orthogonal to the shock. We note also that A' is just the same as S, the area of the shock cut out by the ray tube; A is the normal cross-section.

The differential equation for α is (18) with $\mathbf{i}$ given by (20) and

$$A = \frac{A'}{(1+m^2\alpha_y^2)^{\frac{1}{2}}} = \frac{f(M')}{(1+m^2\alpha_y^2)^{\frac{1}{2}}}; \tag{23}$$

M' is the Mach number of the shock relative to the flow ahead and is given in terms of α by

$$M' = \frac{1 - m\alpha_x}{(\alpha_x^2 + \alpha_y^2)^{\frac{1}{2}}}. \tag{24}$$

The striking difference from Chisnell's attempt is that the rays introduced here are not the orthogonal trajectories of the shock positions. But this is in agreement with the well-known situation in geometrical acoustics, which is, of course, the stimulus for the non-linear theory. It is instructive to see how this theory fits in with geometrical acoustics.

3. Comparison with geometrical acoustics for linear problems

The wave equation for sound waves in a uniform flow U is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{1}{a_0^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\phi. \tag{25}$$

To study discontinuities on $a_0 t = \alpha(x, y)$ we may expand ϕ as

$$\phi = \phi_0(x,y)\,H(a_0 t - \alpha) + \phi_1(x,y)\,H_1(a_0 t - \alpha) + \ldots,$$

where H is the Heaviside step function, H_1 is its integral and so on. This series is substituted in (25) and the coefficients of the singular functions are set equal to zero. The first two terms give

$$E(\alpha_x, \alpha_y) \equiv \alpha_x^2 + \alpha_y^2 - (1 - m\alpha_x)^2 = 0 \tag{26}$$

and
$$2\{m + (1-m^2)\,\alpha_x\}\,\phi_{0x} + 2\alpha_y\,\phi_{0y} + \{(1-m^2)\,\alpha_{xx} + \alpha_{yy}\}\,\phi_0 = 0. \tag{27}$$

The rays are the characteristics for the eikonal equation (26). According to the definition of characteristics, these are curves in the direction of the vector $(\partial E/\partial\alpha_x,\ \partial E/\partial\alpha_y)$. From (26),

$$\frac{\partial E}{\partial \alpha_x} = 2\{m + (1-m^2)\,\alpha_x\}, \quad \frac{\partial E}{\partial \alpha_y} = 2\alpha_y. \tag{28}$$

The magnitude of this vector is the square root of

$$4\{m + (1-m^2)\,\alpha_x\}^2 + 4\alpha_y^2 = 4\{1 + m^2\alpha_y^2\},$$

from (26). The unit ray vector is then

$$\left(\frac{m + (1-m^2)\,\alpha_x}{(1+m^2\alpha_y^2)^{\frac{1}{2}}},\ \frac{\alpha_y}{(1+m^2\alpha_y^2)^{\frac{1}{2}}}\right). \tag{29}$$

It is easily shown through (26) that (20) and (29) agree in this case.

The equation (27) for ϕ_0, the magnitude of the discontinuity, involves the derivative along the ray. Better still, (27) can be written in divergence form as

$$\nabla\,.\,[\mathbf{i}\phi_0^2(1+m^2\alpha_y^2)^{\frac{1}{2}}] = 0, \tag{30}$$

where **i** is the ray vector (29). Appealing to the geometrical result in (17) and (18), we deduce that

$$\phi_0^2 \propto \frac{1}{A(1+m^2\alpha_y^2)^{\frac{1}{2}}}, \tag{31}$$

where A is the cross-sectional area of a slender ray tube. The velocity of the wave front along the ray is

$$C_0 = \frac{a_0}{(\mathbf{i}.\mathbf{n})} = a_0(1+m^2\alpha_y^2)^{\frac{1}{2}},$$

so that (31) can be written as conservation of energy flux

$$\phi_0^2 C_0 A = \text{constant}. \tag{32}$$

Finally, let us collect results and note how the non-linear theory contained in (18), (20), (23) and (24) may be reduced to geometrical acoustics for the linear case. First we set $M' = 1$ in (24). This uncouples the equations and gives immediately an equation for α. It is the correct eikonal equation (26). Then, as we have seen above, (18) and (20) give the correct ray geometry. Finally, since

$$f(M') \propto (M'-1)^{-2} \quad \text{as} \quad M' \to 1,$$

(23) becomes

$$(M'-1)^2 \propto \frac{1}{A(1+m^2\alpha_y^2)^{\frac{1}{2}}}. \tag{33}$$

Since $(M'-1)$ is the strength of the disturbance, this agrees with (31). Thus the extended theory presented here transforms correctly between moving reference frames and fits in perfectly with the corresponding geometrical acoustics for linear wave fronts.

This research was supported by the Office of Naval Research under Contract Nonr 220 (56).

REFERENCES

CHISNELL, R. F. 1965 *J. Fluid Mech.* **22**, 103.
WHITHAM, G. B. 1957 *J. Fluid Mech.* **2**, 145.
WHITHAM, G. B. 1959 *J. Fluid Mech.* **5**, 369.